GCSE Maths – Algebra

Numerical Iteration (Higher Only)

Notes

WORKSHEET

Numerical Iteration

The definition of iterate is to **repeat a process**. When solving an equation, iteration means **substituting** in a number, obtaining the result, then using this result to **substitute in again** to repeat the process.

Iteration is usually performed when we cannot work out the solution to an algebraic equation any other easier way.

The question will often give a guide of the starting value. For example, let's use iteration to work out the following problem:

Find a solution to $x^2 - 3x - 9 = 0$, using a starting value $x_0 = 4.3$. Give the solution to 3 decimal places.

To do this, we first need to rearrange the equation so that we have x on one side as follows:

$$x^2 = 9 + 3x$$

$$x = \sqrt{9 + 3x}$$

To show that we will use iteration, we add subscripts to each x. This indicates which iteration number we are on. E.g. x_1 is the result of the first substitution, x_2 the result of the second…

$$x_{n+1} = \sqrt{9 + 3x_n}$$

Now, substitute x_n for the starting value, which is $x_0 = 4.3$.

$$x_1 = \sqrt{9 + 3 \times 4.3} = 4.680\ (3\ d.p)$$

Then substitute the value $x_1 = 4.680$ in for x_n:

$$x_2 = \sqrt{9 + 3 \times 4.680} = 4.800\ (3\ d.p)$$

Continue to substitute in the answer until we consistently obtain the same number (to 3 d.p).

$$x_3 = \sqrt{9 + 3 \times 4.800} = 4.837\ (3\ d.p)$$
$$x_4 = \sqrt{9 + 3 \times 4.837} = 4.849\ (3\ d.p)$$
$$x_5 = \sqrt{9 + 3 \times 4.849} = 4.853\ (3\ d.p)$$
$$x_6 = \sqrt{9 + 3 \times 4.853} = \mathbf{4.854}\ (3\ d.p)$$
$$x_7 = \sqrt{9 + 3 \times 4.854} = \mathbf{4.854}\ (3\ d.p)$$

Once we get the same value to 3 decimal places, we can stop the iteration process and write the final answer $\boldsymbol{x = 4.854}$.

A quicker way to perform the iteration process on a **calculator** is to use the 'ANS' button in place of x_n. Each time the '=' sign is pressed, the answer will be substituted in. However, you still need to show the **working** and each **result of the iteration**!

Example: Find a solution to the equation $x^3 - 8x - 15 = 0$, using a starting value of $x_0 = 3$. Give the solution to 3 decimal places.

1. Rearrange the equation so that x is on one side.

$$x^3 = 8x + 15$$
$$x = \sqrt[3]{8x + 15}$$

2. Add the subscript notation to show the iterative process.

$$x_{n+1} = \sqrt[3]{8x_n + 15}$$

3. Use the starting value to perform the first iteration to find x_1.

We have been given the starting value $x_0 = 3$, so we substitute that in as x_n:

$$x_1 = \sqrt[3]{8 \times 3 + 15} = 3.391\ (3\ d.p)$$

4. Now substitute the answer in for x_n, and repeat until the answer is the same to 3 decimal places.

$$x_2 = \sqrt[3]{8 \times 3.391 + 15} = 3.480\ (3\ d.p)$$
$$x_3 = \sqrt[3]{8 \times 3.480 + 15} = 3.500\ (3\ d.p)$$
$$x_4 = \sqrt[3]{8 \times 3.5 + 15} = 3.503\ (3\ d.p)$$
$$x_5 = \sqrt[3]{8 \times 3.503 + 15} = \mathbf{3.504}\ (3\ d.p)$$
$$x_6 = \sqrt[3]{8 \times 3.504 + 15} = \mathbf{3.504}\ (3\ d.p)$$

Once we have obtained the same answer twice (to 3 decimal places), we can stop and write the final solution.

$$\boldsymbol{x} = \mathbf{3.504}\ (to\ 3\ d.p)$$

The question may tell us that the solution lies between two numbers, rather than giving us a specific starting value. In this case, we can choose our **own starting value** - either one of the whole numbers, or a decimal in between. It won't make a difference, because we'll get the same answer at the end, it may just take more iterations.

Example: A root of $x^2 - 5x + 1$ lies between 4 and 5.
Using numerical iteration, find the value of the root to 3 decimal places.

1. Rearrange the equation so that x is on one side of the equation.

$$x^2 - 5x + 1 = 0$$

$$x = \sqrt{5x - 1}$$

2. Add in the iteration notation.

$$x_{n+1} = \sqrt{5x_n - 1}$$

3. Choose a starting value for x_0 and perform iterations.

Let's make our starting value $x_0 = 4.5$.

$$x_1 = \sqrt{5 \times 4.5 - 1} = 4.637$$
$$x_2 = \sqrt{5 \times 4.637 - 1} = 4.710$$
$$x_3 = \sqrt{5 \times 4.710 - 1} = 4.749$$
$$x_4 = \sqrt{5 \times 4.749 - 1} = 4.769$$
$$x_5 = \sqrt{5 \times 4.769 - 1} = 4.780$$
$$x_6 = \sqrt{5 \times 4.780 - 1} = 4.785$$
$$x_7 = \sqrt{5 \times 4.785 - 1} = 4.788$$
$$x_8 = \sqrt{5 \times 4.788 - 1} = 4.790$$
$$x_9 = \sqrt{5 \times 4.790 - 1} = \mathbf{4.791}$$
$$x_{10} = \sqrt{5 \times 4.791 - 1} = \mathbf{4.791}$$

Now that we've got the same answer twice, we can stop.

The final solution is $x = \mathbf{4.791}$

Important!
When you substitute previous values of x_n into the equation to find x_{n+1}, make sure you substitute in the full number given on your calculator display. If you substitute in a rounded value, the accuracy will be lost and you may not obtain the correct approximation.

Numerical Iteration – Practice Questions

1. Using numerical iteration, calculate a solution to the following equations. Give the solutions to 3 decimal places.

   a) $x^2 + 3x - 80 = 0$, starting with $x_0 = 7.6$

   b) $2x^3 - 8x^2 - 5 = 0$, with a starting value of $x_0 = 4.1$

   c) $2x^3 + 4x = 14$, with a starting value of $x_0 = 1$

   d) $0.5x^3 + 2.5x - 10 = 0$, with a starting value of $x_0 = 2$

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.